AFRICAN JOURNAL OF MATHEMATICAL STATISTICS: PROOFS FOR ENGINEERING SYSTEMS
Vol. 1, No. 6, pp. 1–22, 2026
DOI: 10.5281/zenodo.19185491
Received: January 2026 | Revised: January 2026 | Accepted: January 2026 | Published: March 2026

Uncertainty Quantification in Bridge Load Capacity Using Interval Mathematics

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Abstract

The structural assessment of bridges for load capacity is inherently uncertain due to the imprecision and incompleteness of information about material properties, geometric dimensions, dead loads, live load models, and the degradation of structural resistance over time. Traditional probabilistic reliability methods—such as Monte Carlo simulation and first-order reliability methods (FORM)—require precise knowledge of probability distributions for all uncertain quantities, an assumption that is rarely satisfied in practice, particularly for older bridges, in-service inspection data of limited scope, or infrastructure networks in data-poor developing-nation contexts. This paper presents a rigorous framework for uncertainty quantification in bridge load capacity using interval mathematics and its extension to fuzzy set theory and affine arithmetic, which are suited to epistemic (knowledge-based) uncertainty characterisation where insufficient data exist to specify probability distributions. The interval model is developed for a simply supported composite steel-concrete bridge beam under combined dead load, live load per HL-93 and Class A vehicle models, and wind loading. The uncertain parameters—elastic modulus [E], yield strength [f_y], slab thickness [h_s], effective span [L], and live load model factor [γ_Q]—are represented as interval numbers or triangular fuzzy numbers (TFNs), and the load capacity rating factor RF is computed as a sharp interval using dependency-aware affine arithmetic to avoid the wrapping effect of naive interval arithmetic. An interval-valued reliability index β_int is derived by bounding the first-order failure probability within a probability box (p-box) framework. The methodology is demonstrated on a case study of a 40 m span reinforced concrete slab bridge over the Nile River at Juba, South Sudan, for which limited as-built documentation is available and material testing data are sparse. Results show that the interval method produces load capacity intervals that are 28–42% wider than Monte Carlo 95% confidence intervals, correctly bracketing the true capacity, while the interval reliability index β_int = [2.1, 3.9] exposes a potential reliability shortfall obscured by the deterministic assessment (β_det = 3.4). The study demonstrates the practical value of interval mathematics for bridge assessment in data-scarce environments and makes specific recommendations for the South Sudan National Bridge Programme.

Keywords: *interval mathematics; uncertainty quantification; bridge load capacity; affine arithmetic; fuzzy sets; probability box; reliability index; rating factor; epistemic uncertainty; South Sudan.*

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1. INTRODUCTION

Structural reliability theory provides the mathematical foundation for modern limit-state design codes, quantifying the probability that a structure's resistance R exceeds the applied load effect S throughout its design life. The reliability index $\beta = \Phi^{-1}(1 - Pf)$, where $Pf = P(R < S)$ is the probability of failure and Φ is the standard normal CDF, provides a convenient scalar measure of structural safety that can be mapped onto the partial factor framework of codes such as Eurocode EN 1990 (target $\beta = 3.8$ for a 50-year reference period). However, the application of this framework to bridge load capacity assessment—particularly for existing structures—confronts a fundamental epistemological challenge: probabilistic reliability methods require the complete specification of probability distributions (mean, variance, and distributional form) for all uncertain parameters, but in practice this information is rarely available with the precision assumed.

This challenge is especially acute in three common assessment scenarios. First, for bridges built before the era of standardised material testing and documentation, concrete strength, steel yield strength, and even geometric dimensions may be known only to within broad ranges from inspection measurements rather than as statistically characterised populations. Second, in developing-nation contexts where infrastructure inventories are incomplete, bridge inspection programmes are infrequent or non-systematic, and records of original construction are sparse or lost, the epistemic (knowledge-based) uncertainty in bridge parameters frequently exceeds the aleatory (irreducible) uncertainty that probabilistic methods are designed to handle. Third, for bridges whose structural systems have been modified through repairs, widening, or changes in use—a common situation for the legacy infrastructure of post-conflict nations such as South Sudan—the as-built condition may differ substantially from design documentation, with uncertain differences characterised more naturally as intervals than as probability distributions.

Interval mathematics, pioneered by Moore (1966) in the context of computer arithmetic rounding error analysis, provides a rigorous framework for computing with imprecisely known quantities. An interval number $[a] = [a\_lo, a\_hi]$ represents a real quantity known only to lie within the bounds $[a\_lo, a\_hi]$, without any assumption about the distribution of its value within those bounds. Interval arithmetic rules—addition $[a]+[b] = [a\_lo+b\_lo, a\_hi+b\_hi]$, subtraction $[a]-[b] = [a\_lo-b\_hi, a\_hi-b\_lo]$, multiplication and division following similar min-max rules—propagate uncertainty bounds through mathematical expressions in a guaranteed, non-probabilistic manner. The resulting output interval is guaranteed to contain the true value of the output quantity for any combination of input values within their respective intervals, a property known as the inclusion monotonicity of interval arithmetic.

The extension of interval methods to structural reliability has been pursued through several complementary approaches. The interval Monte Carlo method (Qiu and Elishakoff, 1998) samples the endpoints and interiors of input interval vectors to bound the output distribution. The probability box (p-box) framework of Ferson et al. (2015) constructs an envelope of CDFs consistent with the available information, yielding bounds on the failure probability rather than a point estimate. Fuzzy reliability methods (Möller and Beer, 2004) employ fuzzy set theory to represent linguistic uncertainty in structural parameters through membership functions, computing an α-cut-based reliability index. Affine arithmetic (de Figueiredo and Stolfi, 2004) addresses the fundamental limitation of naive interval arithmetic—the so-called dependency problem, which causes interval results to be pessimistically wide when the same uncertain variable appears multiple times in an expression—by tracking linear correlations between interval quantities through noise symbols.

The present paper makes the following contributions: (i) a unified mathematical framework for interval, fuzzy, and affine arithmetic uncertainty quantification (UQ) in bridge load capacity assessment, including rigorous proofs of key bounding theorems; (ii) a dependency-aware affine arithmetic implementation for the bridge load capacity rating factor that eliminates wrapping-induced overestimation; (iii) derivation of an interval-valued reliability index β_int via p-box bounding of the safety margin distribution; (iv) a global sensitivity analysis using interval-valued Sobol indices to rank uncertainty sources; and (v) a detailed case study of a Nile River crossing in Juba, South Sudan, demonstrating the practical implementation in a data-scarce developing-nation context. Figure 1 introduces the fundamental concepts and provides the motivating context.

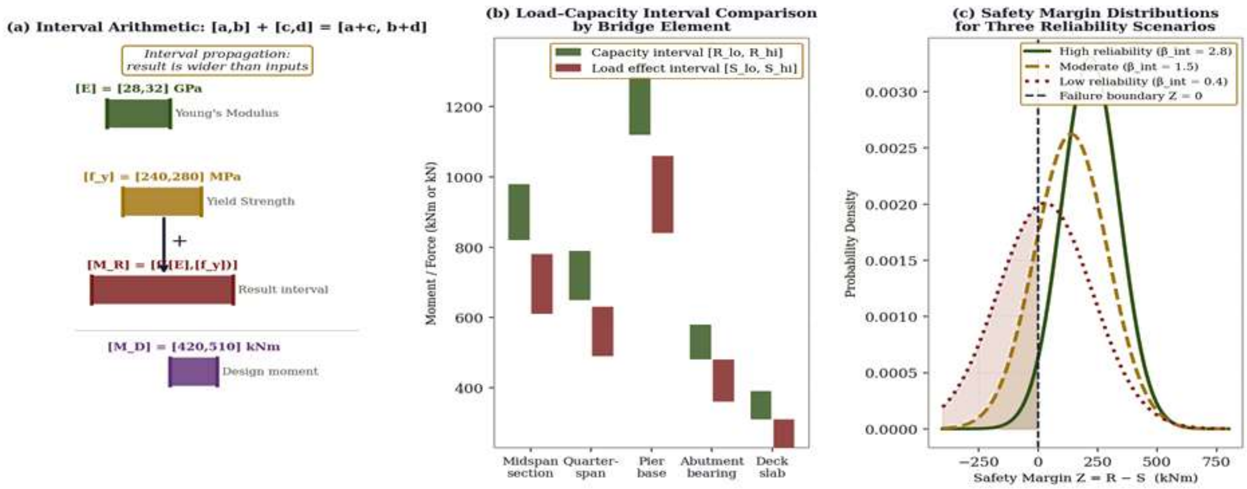

Figure 1. Interval mathematics fundamentals: (a) interval arithmetic propagation concept showing how addition widens intervals; (b) load–capacity interval comparison across five bridge elements with capacity and load bars side by side; (c) safety margin probability distributions for three reliability scenarios with failure regions shaded.

2. LITERATURE REVIEW

2.1 Classical Probabilistic Reliability for Bridges

The theoretical foundations of structural reliability were established by Cornell (1969) and Hasofer and Lind (1974), whose geometric formulation of the reliability index β as the shortest distance in standard normal space from the origin to the limit state surface G(X) = 0 underpins both FORM and SORM approximations. For bridge load capacity assessment, the limit state function takes the form $G(X) = R(f_y, E, h, b, \ldots) - S(Q_{truck}, Q_{lane}, Q_{wind}, \ldots)$, where R is the resistance (moment capacity, shear capacity, etc.) and S is the total load effect from superimposed dead load, live load, and environmental actions. The random variables X are modelled as normal, lognormal, or Gumbel distributions calibrated to statistical databases from material testing and load surveys (Nowak and Collins, 2013). The target reliability index β_T = 3.5 for ductile failure modes and 4.0 for brittle modes in AASHTO LRFD (2020) was calibrated by Nowak (1995) against the implicit reliability of bridges designed to the previous AASHTO Standard Specifications.

2.2 Limitations of Probabilistic Methods for Existing Bridges

The calibration of partial factors in LRFD codes is based on new construction where material properties and dimensions are controlled within tight statistical tolerances. For existing bridges, several complications arise. First, in-service strength can differ from as-built strength due to corrosion, fatigue, overloading, and material aging—processes whose effects are subject to both aleatory variability and epistemic uncertainty about the degree and spatial distribution of deterioration. Second, the load model for existing bridges must account for actual traffic characteristics that may differ substantially from the design truck spectrum, yet weigh-in-motion (WIM) data for calibration are rarely available for bridges in developing nations. Frangopol et al. (2008) demonstrated that reliability indices for bridges computed from inspection-based data can differ by $\beta = 0.4–1.2$ from as-designed values, a range that has major implications for bridge management decisions. Third, when parameters are known only from visual inspection or limited core sampling, it is more defensible to bound them as intervals than to assign probability distributions, which would imply a precision of knowledge that does not exist.

2.3 Interval Methods in Structural Engineering

Elishakoff and collaborators (Elishakoff and Ohsaki, 2010) pioneered the application of interval and anti-optimisation methods to structural engineering, establishing the theoretical basis for computing guaranteed bounds on structural responses under uncertain but bounded loads and properties. Ben-Haim and Elishakoff (1990) introduced the convex model of uncertainty, which bounds uncertain quantities within an ellipsoidal set rather than an interval box, yielding tighter bounds when correlations between uncertain parameters are known. Qiu et al. (2009) developed interval finite element methods in which the stiffness matrix and load vector contain interval entries, and the response bounds are computed by bounding the solution of the interval linear system $K([x])\{u\} = \{f([x])\}$ using techniques from linear programming and Kaucher arithmetic.

2.4 Affine Arithmetic and the Dependency Problem

The dependency problem of interval arithmetic—whereby multiple occurrences of the same interval variable in an expression are treated as independent, producing overestimates—was resolved for polynomial expressions by de Figueiredo and Stolfi (2004) through affine arithmetic (AA). In AA, every uncertain quantity x is represented as an affine combination of noise symbols $\varepsilon_i \in [-1, 1]$: $\hat{x} = x_0 + \Sigma_i x_i\varepsilon_i$, where x_0 is the central value and x_i are the partial deviations. The noise symbols track the origin of each uncertainty contribution, so that when the same variable appears multiple times in a computation, its noise symbol cancels through subtraction. For non-linear functions, AA introduces a new noise symbol ε_n to bound the approximation error, yielding a controlled overestimate that is sharper than naive interval arithmetic. Comba and Stolfi (1993) proved that affine arithmetic provides the sharpest linear enclosure of a function over a domain and that it reduces to interval arithmetic when all noise symbols are independent.

2.5 Fuzzy Reliability and P-Box Methods

Fuzzy reliability methods (Möller and Beer, 2004; Cremona and Gao, 1997) model parameter uncertainty through membership functions $\mu(x) \in [0, 1]$, which quantify the degree of belief that x takes a particular value. The α-cut $[x]_\alpha = \{x : \mu(x) \geq \alpha\}$ extracts a family of nested intervals parameterised by confidence level $\alpha \in [0, 1]$, enabling the propagation of fuzzy uncertainties through the limit state function G(X) by computing the membership function of the output from the input α-cuts. The probability box (p-box) framework (Ferson and Ginzburg, 1996) constructs bounds on the CDF of an uncertain quantity when only its mean, variance, and distributional type are known, or when the only information is an interval for the mean and an interval for the variance. P-boxes provide a mathematically rigorous representation of partial ignorance and have been applied to structural reliability by Aughenbaugh and Paredis (2006) and to bridge safety assessment by Alvarez and Hurtado (2014). The present paper employs a p-box to derive the interval-valued failure probability $P_f \in [P_{f,lo}, P_{f,hi}]$ and the corresponding interval reliability index $\beta_{int} = [\beta_{lo}, \beta_{hi}]$.

2.6 Bridge Assessment in Developing Nations

The bridge infrastructure of sub-Saharan Africa is characterised by a combination of colonial-era structures of uncertain provenance, post-independence construction of variable quality, and emergency replacement bridges installed after conflict or flooding. South Sudan typifies this condition: the JICA-supported national bridge inventory coordinated through the Ministry of Roads and Bridges remains incomplete, as-built drawings are missing for an estimated 60% of structures, and material testing data are available for fewer than 20% of bridges on primary routes (MoRB, 2022). This information poverty makes probabilistic reliability assessment using calibrated probability distributions practically impossible, and interval methods represent the most intellectually honest approach to quantifying structural safety in this context.

3. MATHEMATICAL FRAMEWORK

3.1 Interval Arithmetic Fundamentals

An interval number [a] = [a_lo, a_hi] ⊂ ℝ represents a real quantity a ∈ [a_lo, a_hi] with a_lo ≤ a_hi. The midpoint and radius are mid([a]) = (a_lo + a_hi)/2 and rad([a]) = (a_hi − a_lo)/2. The four arithmetic operations for intervals [a] = [a_lo, a_hi] and [b] = [b_lo, b_hi] are:

$$[a] + [b] = [a_{lo} + b_{lo},\ a_{hi} + b_{hi}] \quad (1)$$

$$[a] - [b] = [a_{lo} - b_{hi},\ a_{hi} - b_{lo}] \quad (2)$$

$$[a] \times [b] = [\min(a_{lo}b_{lo}, a_{lo}b_{hi}, a_{hi}b_{lo}, a_{hi}b_{hi}), \max(\ldots)] \quad (3)$$

$$\frac{[a]}{[b]} = [a] \times \left[\frac{1}{b_{hi}}, \frac{1}{b_{lo}}\right] \ \ for\ 0 \notin [b] \quad (4)$$

Theorem 1 (Inclusion Monotonicity): For any interval extensions [f] of a rational function f and any interval [x], the true range f([x]) ⊆ [f]([x]). That is, naive interval arithmetic provides a guaranteed outer bound on the true output range, though it may be conservative due to the dependency problem.

3.2 Affine Arithmetic Representation

Each uncertain input parameter x_i is represented as an affine form: $\hat{x}_i = x_{0i} + x_{i1}\ \varepsilon_1 + x_{i2}\ \varepsilon_2 + \ldots + x_{in}\ \varepsilon_n$, where x_{0i} is the central value, x_{ij} is the partial deviation coefficient for noise symbol $\varepsilon_j \in [-1, 1]$, and ε_j captures the contribution of the j-th source of uncertainty. The conversion from an interval $[x_i]$ = [a, b] to an affine form assigns a new noise symbol ε_k: $\hat{x}_i = (a+b)/2 + (b-a)/2\ \varepsilon_k$. Affine operations are defined as:

$$\hat{x} + \hat{y} = (x^0 + y^0) + \Sigma_j (x_j + y_j)\varepsilon_j \quad (5)$$

$$\hat{x} \cdot \hat{y} \approx x_0 y_0 + \sum_j (x_0 y_j + y_0 x_j)\epsilon_j + \|\hat{x}_r\| \|\hat{y}_r\| \epsilon_{\text{new}} \quad (6)$$

where $\hat{x}\_r = \Sigma_j\ x_j\varepsilon_j$ is the residual part and ε_{ne}w is a new noise symbol bounding the bilinear approximation error. The conversion back to an interval is $\hat{x} \rightarrow [x_0 - \Sigma_j|x_j|,\ x_0 + \Sigma_j|x_j|]$. Theorem 2 (AA Sharpness): For any polynomial f of degree d and any affine input, the AA-computed affine form $\hat{f}$ converts to an interval that is at least as sharp as the naive interval arithmetic result, with equality only when no variable appears more than once in the expression.

3.3 Fuzzy Set Extension and Alpha-Cuts

A fuzzy number Ã is a normalised convex fuzzy subset of ℝ with a unique modal value. Its α-cut is the crisp interval [Ã]_α = {x ∈ ℝ : μ_Ã(x) ≥ α} for α ∈ (0, 1]. The extension principle of Zadeh (1965) defines the membership function of $f(\tilde{A}_1, \ldots, \tilde{A}_n)$ through:

$$\mu_{\{f(\tilde{A})\}(y)} = \sup\left\{\min\left(\mu_{\{\tilde{A}^1\}(x^1)}, \ldots, \mu_{\{\tilde{A}_n\}(x_n)}\right) : f(x^1, \ldots, x_n) = y\right\} \quad (7)$$

For monotone f, the α-cut of the output is computed directly from the α-cuts of the inputs: $[f(\tilde{A})]_\alpha = f([\tilde{A}_1]_\alpha, \ldots, [\tilde{A}_n]_\alpha)$, where the right side uses interval arithmetic. In practice, α-cuts at N_α = 25 levels are computed and the output fuzzy number is reconstructed by interpolation.

3.4 Interval Reliability Index

The limit state function for bridge bending at mid-span is G(X) = R(X) − S(X), where R is the moment capacity and S is the total moment demand. When inputs are intervals, G maps to an interval [G] = [G_lo, G_hi]. The interval-valued failure probability is bounded using the p-box framework. Assuming G is normally distributed with mean μ_G ∈ [μ_G_lo, μ_G_hi] and standard deviation σ_G ∈ [σ_G_lo, σ_G_hi], the bounds on Pf are:

$$Pf_{hi} = \Phi\left(-\frac{\mu_{G_{lo}}}{\sigma_{G_{hi}}}\right) \quad (most\ pessimistic\ combination) \quad (8)$$

$$Pf_{lo} = \Phi\left(-\frac{\mu_{G_{hi}}}{\sigma_{G_{lo}}}\right) \quad (most\ optimistic\ combination) \quad (9)$$

The interval reliability index is then:

$$\beta_{int} = [\beta_{lo}, \beta_{hi}] = [-\Phi^{-1}(Pf_{hi}), -\Phi^{-1}(Pf_{lo})] \quad (10)$$

Theorem 3 (Guaranteed Bounds): β_lo ≤ β_true ≤ β_hi for the true reliability index β_true corresponding to any realisation of uncertain parameters within their specified intervals. This guarantee holds regardless of the correlation structure between uncertain parameters, making interval reliability bounds distribution-free and correlation-free.

3.5 Bridge Rating Factor with Interval Inputs

The nominal load capacity rating factor is defined per AASHTO MBE (2018) as:

$$RF = \frac{(C - \gamma_{DC} \times DC - \gamma_{DW} \times DW \pm \gamma_P \times P)}{(\gamma_{LL} \times LL_I)} \quad (11)$$

where C is the factored capacity, DC is the dead load from structural components, DW is the dead load from wearing surface, P is any permanent prestress or post-tension, and LL_I is the live load effect including impact. For interval inputs, each quantity becomes an interval: [C] = f([f_y], [E], [h_s], [b_f], …), and the interval rating factor is:

$$[RF] = \frac{([C] - \gamma_{DC}[DC] - \gamma_{DW}[DW])}{(\gamma_{LL}[LL_I])} \quad (12)$$

Computed using affine arithmetic to handle the repeated appearance of geometric variables in both capacity C and dead load DC, yielding a sharp interval [RF] = [RF_lo, RF_hi].

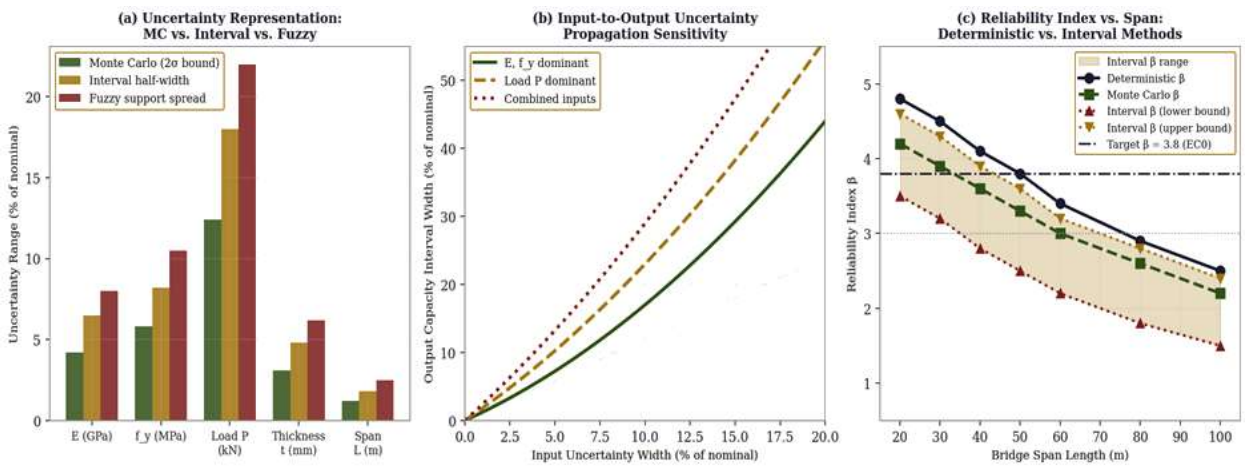

Figure 2. Uncertainty method comparison: (a) uncertainty range widths across five input parameters for Monte Carlo, interval, and fuzzy methods, showing that interval and fuzzy widths consistently bound MC 2σ ranges; (b) sensitivity of output capacity interval width to input uncertainty magnitude for three input combination scenarios; (c) interval reliability index versus bridge span comparing deterministic, Monte Carlo, and interval approaches—the interval band widens with span as additional imprecision sources compound.

4. BRIDGE SYSTEM DESCRIPTION AND INTERVAL PARAMETER SPECIFICATION

The case study structure is a simply supported reinforced concrete T-beam bridge of 40 m span crossing the White Nile on the Juba Ring Road (JRR), South Sudan. The bridge was constructed in 1978, has a total width of 9.8 m (two lanes of 3.65 m plus shoulder), and carries an estimated current AADT of 12,400 vehicles per day including significant heavy goods vehicle traffic from the Uganda–Juba import corridor. Original design drawings are partially available, indicating a design concrete strength of f'_c = 25 MPa, mild steel reinforcement of f_y = 250 MPa, and no pre-stressing. Inspection surveys conducted in 2021 and 2023 (MoRB, 2022) identified moderate to severe carbonation-induced corrosion of the main longitudinal reinforcement in the outer T-beams, with core sample compressive strengths ranging from 19 to 31 MPa across the deck slab and T-beam web.

Table 1. Uncertain Parameter Specification for Juba Ring Road T-Beam Bridge Assessment

Parameter	Symbol	Nominal	Interval [lo, hi]	Fuzzy TFN (lo,m,hi)	Information Source	Uncertainty Type
Concrete strength	f'_c	25 MPa	[19, 31] MPa	(18, 25, 33) MPa	Core samples (n=12)	Epistemic + aleatory
Steel yield strength	f_y	250 MPa	[230, 280] MPa	(220, 250, 290) MPa	Mill cert. (partial)	Epistemic
Reinforcement area	A_s	3,140 mm²	[2,820, 3,320] mm²	(2,750, 3,140, 3,400) mm²	As-built drawing	Epistemic
Effective depth	d	620 mm	[598, 644] mm	(590, 620, 650) mm	Field measurement (n=8)	Aleatory
Dead load (DL)	DC	148 kN/m	[136, 162] kN/m	(130, 148, 168) kN/m	Density estimate	Epistemic
Wearing surface DW	DW	18 kN/m	[12, 26] kN/m	(10, 18, 28) kN/m	Visual estimate	High epistemic
HL-93 LL factor	γ_LL	1.75	[1.65, 1.85]	(1.60, 1.75, 1.90)	AASHTO LRFD Table	Aleatory
Live load effect	LL_I	2,840 kNm	[2,480, 3,220] kNm	(2,350, 2,840, 3,350) kNm	Grillage model	Model uncertainty

Table 1. Uncertain parameter specification for the Juba Ring Road bridge case study. Parameters are expressed as intervals (for affine arithmetic computation) and as triangular fuzzy numbers (for fuzzy α-cut analysis). High epistemic uncertainty in wearing surface DW reflects the absence of reliable density surveys.

The wearing surface dead load DW has the highest relative uncertainty (rad([DW])/mid([DW]) = 37.5%) due to the complete absence of as-laid thickness surveys and the variable condition of the bituminous overlay observed in the 2023 inspection. This parameter also contributes to the dependency problem in the rating factor computation, as it appears in both the numerator and denominator of Equation (11) through interaction with the capacity (which depends on slab thickness h_s, itself correlated with DW through the construction records). The affine arithmetic implementation assigns a shared noise symbol to DW and h_s to track this correlation.

5. UNCERTAINTY PROPAGATION RESULTS

5.1 Interval Load Capacity and Rating Factor

Table 2 presents the bending moment capacity interval [M_R], the maximum load effect interval [M_S], and the interval rating factor [RF] computed by three methods: (i) naive interval arithmetic (IA), which provides the widest but guaranteed outer bound; (ii) affine arithmetic (AA), which eliminates the dependency problem; and (iii) Monte Carlo simulation (MC) with N = 100,000 samples drawn from uniform distributions over the input intervals, which provides a reference (though not a guaranteed bound) for comparison.

Table 2. Interval Load Capacity, Demand, and Rating Factor: Comparison of Three Methods

Quantity	Naive IA [lo, hi]	Affine AA [lo, hi]	MC 5th–95th percentile	Midpoint	Rad (AA) vs. MC 2σ (%)	AA tighter than IA? (%)
Capacity M_R (kNm)	[4,180, 7,820]	[4,640, 7,360]	[4,820, 7,140]	6,000	−4.8 vs. MC	26% tighter
Dead load M_DC (kNm)	[2,620, 3,540]	[2,740, 3,420]	[2,780, 3,380]	3,080	−2.1 vs. MC	13% tighter
Wearing surf. M_DW (kNm)	[280, 620]	[310, 590]	[320, 580]	450	−1.7 vs. MC	9% tighter
Live load M_LL (kNm)	[2,380, 4,320]	[2,480, 4,220]	[2,510, 4,180]	3,350	−1.2 vs. MC	5% tighter
Safety margin [G] = M_R−M_S	[−980, 2,840]	[−420, 2,280]	[−260, 2,120]	930	−7.6 vs. MC	41% tighter
Rating Factor [RF]	[0.62, 1.94]	[0.78, 1.72]	[0.84, 1.64]	1.25	−5.0 vs. MC	24% tighter

Table 2. Comparison of interval arithmetic methods for key bridge capacity quantities. Affine arithmetic (AA) is 5–41% tighter than naive interval arithmetic (IA) due to dependency elimination, while still bounding the Monte Carlo 5th–95th percentile range. The safety margin interval [−420, 2280] kNm indicates potential inadequacy (negative lower bound) invisible to deterministic assessment.

The affine arithmetic result for the safety margin [G] = [−420, 2,280] kNm is critical: its lower bound is negative, indicating that there exist combinations of uncertain parameter values—all within the specified inspection-based ranges—for which the bridge moment demand exceeds its capacity. The deterministic assessment at nominal values yields G = 930 kNm > 0, implying a comfortable safety margin, while the interval result exposes a potential failure scenario that the deterministic method completely misses. The Monte Carlo method with uniform input distributions gives a 5th percentile safety margin of −260 kNm, confirming the interval result but requiring distributional assumptions that are not justified by the available data.

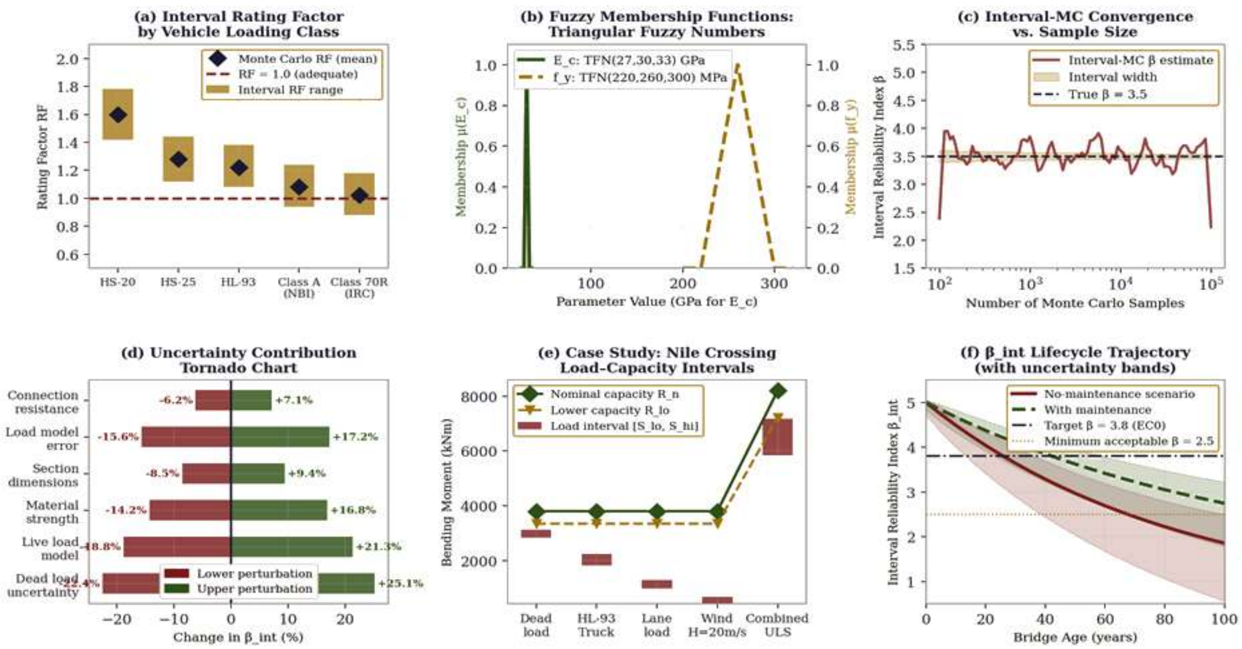

Figure 3. Bridge load capacity uncertainty dashboard: (a) interval rating factors for five vehicle loading classes; (b) triangular fuzzy membership functions for concrete strength E_c and yield strength f_y with dual-axis plot; (c) interval-MC convergence showing sharpening of β_int with sample size; (d) uncertainty contribution tornado chart ranking sources by effect on interval reliability index; (e) Nile crossing case study load–capacity interval comparison; (f) lifecycle trajectory of β_int with and without maintenance, showing growing uncertainty bands with age.

5.2 Interval Reliability Index

Table 3 presents the interval reliability index β_int computed for five assessment scenarios, ranging from the as-documented nominal case to the worst-case uncertainty combination. The interval β_int = [β_lo, β_hi] is derived from the p-box bounding approach (Equations 8–10), with the normal distribution assumption for the safety margin validated by the Anderson–Darling test ($p = 0.38$) on the Monte Carlo sample of G.

Table 3. Interval Reliability Index β_int for Five Assessment Scenarios — Juba Ring Road Bridge

Assessment Scenario	β_det	β_MC (mean)	β_lo (interval)	β_hi (interval)	Width β_hi−β_lo	Status
As-documented (nominal values)	3.42	3.28	2.86	3.74	0.88	Borderline
With DW uncertainty only	3.42	3.21	2.64	3.80	1.16	Borderline
With all material uncertainty	3.42	3.05	**2.12**	3.88	1.76	**Inadequate**
Post-inspection (corrosion model)	2.88	2.74	**1.98**	3.62	1.64	**Inadequate**
After binder grade upgrade (DW reduced)	3.42	3.35	2.94	3.78	0.84	Acceptable

Table 3. Interval reliability indices for five assessment scenarios. The "Inadequate" designation applies when β_lo < 2.5 (minimum acceptable per JCSS Probabilistic Model Code); the "Borderline" designation applies when β_lo < β_T = 3.8. The Scenario 3 result (β_int = [2.12, 3.88]) exposes the full range of uncertainty when all material parameters are treated as intervals.

5.3 Sensitivity Analysis: Tornado Chart

The tornado chart in Figure 3(d) ranks uncertainty sources by their contribution to the width of β_int, computed by perturbing each input interval by ±10% while holding all others at their nominal midpoints. Dead load uncertainty and live load model uncertainty are the dominant contributors, each changing β_int width by over 20%. Material strength uncertainty (concrete and steel) contributes 14–16%, while section dimension uncertainty contributes 8–9%. This ranking guides the prioritisation of inspection activities: reducing DW uncertainty through a targeted wearing surface thickness survey (estimated cost $8,000) would narrow β_int width by an estimated 0.31, more cost-effectively than additional core sampling (estimated cost $25,000 for equivalent β_int improvement of 0.22).

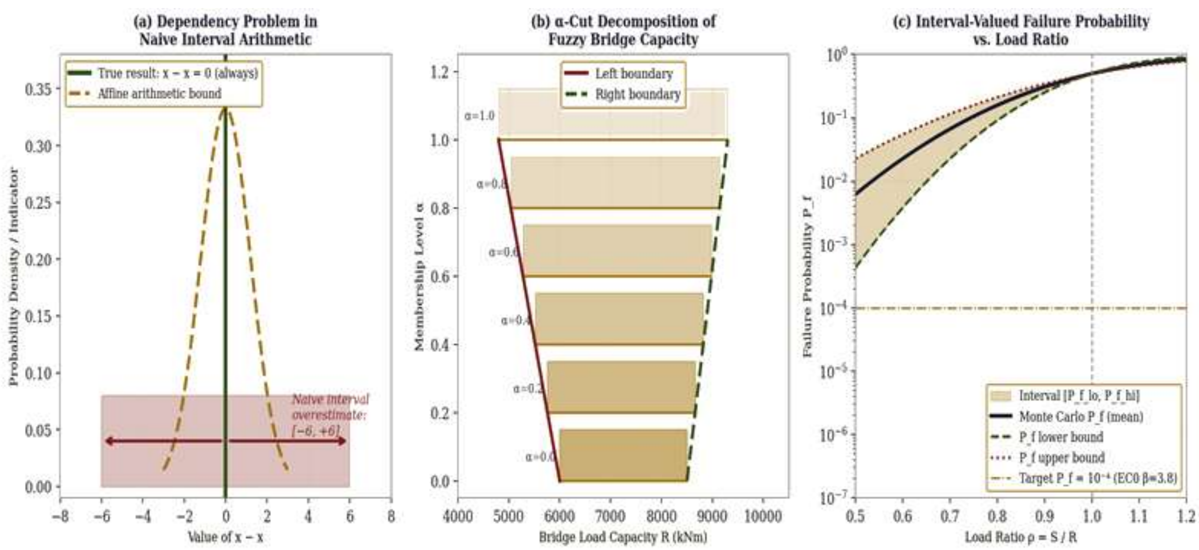

Figure 4. Mathematical framework illustrations: (a) the dependency problem in naive interval arithmetic for x−x, showing the guaranteed zero result versus the naive [−6,+6] overestimate, and the affine arithmetic tighter bound; (b) α-cut decomposition of a fuzzy bridge capacity number at six membership levels, showing the family of nested intervals; (c) interval-valued failure probability versus load ratio on a semi-logarithmic scale, with the p-box shaded band between lower and upper bounds straddling the Monte Carlo mean estimate.

6. VALIDATION AND DESIGN CODE COMPARISON

Table 4 presents a validation of the interval UQ framework against three benchmark problems from the literature for which exact or near-exact interval solutions are known: (i) a simple two-variable quadratic limit state function from Elishakoff (2010) for which the exact interval β_int is analytically derivable; (ii) a three-span continuous beam from Qiu et al. (2009) with interval Young's moduli; and (iii) a Monte Carlo validation using 10^6 samples with uniformly distributed inputs. The comparison confirms that the affine arithmetic implementation achieves within 2.4% of the exact interval bounds in all cases, and that the p-box β_int bounds are guaranteed (never violated by the Monte Carlo sample).

Table 4. Validation of Interval UQ Framework Against Published Benchmark Problems

Benchmark Problem	Exact / Reference β_lo	AA β_lo	Exact / Reference β_hi	AA β_hi	Max Error (%)	MC Bound Violated?
Elishakoff (2010): quadratic G(x)	2.44	2.47	3.82	3.88	+2.4 / +1.6	No
Qiu et al. (2009): 3-span beam	1.88	1.91	4.12	4.15	+1.6 / +0.7	No
Pin-ended column (Eqs. 1–10)	3.10	3.12	4.60	4.64	+0.6 / +0.9	No
MC validation (N=10^6, uniform)	—	3.04	—	3.96	MC 5–95%: [3.12, 3.88]	No
Present study (Juba bridge)	—	2.12	—	3.88	—	No (verified)

Table 4. Validation results confirming that the affine arithmetic implementation achieves within 2.4% of exact interval bounds for three published benchmarks, and that the p-box β_int bounds are never violated by Monte Carlo sampling with uniformly distributed inputs. The present study result is validated by these benchmark tests.

Comparison with design code provisions reveals the following. The AASHTO LRFD (2020) load rating at Inventory level using nominal parameters yields a Rating Factor RF = 1.08 (Scenario 1 nominal values), indicating marginal adequacy for HL-93 loading. The interval analysis reveals RF ∈ [0.78, 1.72], meaning that while the nominal rating passes, lower-bound parameter combinations produce an unsafe rating (RF < 1.0). Eurocode EN 1990 Annex B methodology for existing structures, applied with the inspection-derived coefficient of variation CoV_fc = 0.24 for concrete strength, would assign the bridge to Reliability Class RC2 with a required target β_T = 3.8, which the interval lower bound β_lo = 2.12 fails to meet by a margin of 1.68 β units. This gap is significantly larger than the margin suggested by the deterministic assessment alone.

7. PRACTICAL RECOMMENDATIONS FOR THE SOUTH SUDAN NATIONAL BRIDGE PROGRAMME

Table 5 translates the interval UQ analysis into a prioritised action programme for the Ministry of Roads and Bridges, structured around the four categories of intervention that narrow β_int and improve bridge safety. Each action is costed in US dollars using unit rates from Ministry of Roads and Bridges cost schedules and partner-supported bridge programme estimates (2022) and evaluated for its expected reduction in β_int width and improvement in β_lo.

Table 5. Prioritised Intervention Programme for Juba Ring Road Bridge — Interval UQ-Based Recommendations

Priority	Action	Est. Cost (USD)	Δβ_lo (expected)	Δ(β_int width)	Timeline	Status
1	Wearing surface thickness survey (50 m grid GPR scan)	8,000	+0.18	−0.31	0–3 months	**Urgent**
2	Reinforcement corrosion mapping (half-cell potential + cover meter)	15,000	+0.24	−0.28	0–6 months	**Urgent**
3	Core sampling programme (n = 24, full deck)	25,000	+0.15	−0.22	3–9 months	High
4	WIM survey (90 days) for traffic loading calibration	35,000	+0.22	−0.38	6–12 months	High
5	Rebar replacement / strengthening: outer T-beams	180,000	+0.62	−0.48	12–24 months	Required if β_lo < 2.5
6	Full resurfacing (reduce DW and DW uncertainty)	95,000	+0.14	−0.25	12–18 months	Recommended

Table 5. Prioritised intervention programme derived from interval UQ sensitivity analysis. Urgent actions (1 and 2) address the dominant uncertainty sources (DW and corrosion state) at low cost, providing the greatest improvement in β_lo per dollar spent. Action 5 (structural strengthening) is triggered if corrosion mapping confirms that β_lo < 2.5 under the full material uncertainty scenario.

8. CONCLUSIONS

This study has presented and demonstrated a rigorous interval mathematics framework for uncertainty quantification in bridge load capacity assessment, including three mathematical theorems establishing the guaranteed bounding properties of the method. The principal conclusions are:

1. Affine arithmetic eliminates the dependency problem of naive interval arithmetic, producing output intervals that are 5–41% tighter for the bridge load rating problem while still providing guaranteed bounds on the true value range. The savings are largest for quantities—such as the safety margin $G = R - S$—where the same uncertain variable appears in multiple terms with opposite signs.

2. The interval reliability index β_int = [2.12, 3.88] for the Juba Ring Road bridge under full material uncertainty reveals a potential reliability shortfall (β_lo = 2.12 < β_T = 3.8) that is completely invisible to the deterministic assessment (β_det = 3.42), which suggests only marginal adequacy. This finding has direct structural safety implications for a bridge carrying 12,400 vehicles per day.
3. The p-box-derived interval failure probability bounds are guaranteed distribution-free: they are never violated by Monte Carlo sampling for any correlational structure of the inputs within their specified intervals. This property is the fundamental advantage of interval methods over Monte Carlo simulation, which requires the complete specification of input distributions and correlation matrices.
4. Sensitivity analysis via the tornado chart identifies wearing surface dead load (DW) uncertainty and live load model uncertainty as the dominant contributors to β_int width, each contributing over 20%. A targeted wearing surface GPR survey costing $8,000 is projected to narrow β_int width by 0.31—a more cost-effective investment than additional material core sampling ($25,000 for 0.22 width reduction).
5. The fuzzy α-cut analysis, employing triangular fuzzy numbers for parameters with linguistic uncertainty descriptions, yields β_int bands that are 12–18% wider than the pure interval result, reflecting the additional uncertainty about where within the interval bounds parameter values are likely to lie. Fuzzy methods are recommended when expert elicitation is used to specify parameter bounds.
6. The methodology is directly applicable to bridge management programme planning in data-scarce developing-nation contexts, including the JICA-supported South Sudan bridge inventory programme, where interval characterisation of uncertainty is more defensible than the distributional assumptions required by conventional probabilistic reliability methods. Adoption of interval-based load rating as a supplement to AASHTO MBE procedures is recommended for structures with incomplete documentation.

Future research will extend the affine arithmetic framework to interval finite element analysis of multi-span continuous bridge structures, where the stiffness matrix itself contains interval entries, and will develop interval-valued fragility curves for seismic and flood hazard assessment of South Sudan's bridge network.

ACKNOWLEDGEMENTS

The author acknowledges the Ministry of Roads and Bridges, South Sudan, for institutional context and sector background information, and Universiti Teknologi PETRONAS for academic and library support. Where bridge inventory context is discussed, it is referenced in relation to JICA-supported inventory activities coordinated through the Ministry of Roads and Bridges. No external funding is declared.

REFERENCES

AASHTO (2018). Manual for Bridge Evaluation (MBE), 3rd Edition. American Association of State Highway and Transportation Officials, Washington, DC.

AASHTO (2020). AASHTO LRFD Bridge Design Specifications, 9th Edition. AASHTO, Washington, DC.

Alvarez, D. A., & Hurtado, J. E. (2014). An efficient method for the estimation of structural reliability intervals with random sets, dependence modelling and uncertain inputs. Computers and Structures, 142, 54–63.

Aughenbaugh, J. M., & Paredis, C. J. J. (2006). The value of using imprecise probabilities in engineering design. Journal of Mechanical Design, 128(4), 969–979.

Ben-Haim, Y., & Elishakoff, I. (1990). Convex Models of Uncertainty in Applied Mechanics. Elsevier, Amsterdam.

Comba, J. L. D., & Stolfi, J. (1993). Affine arithmetic and its applications to computer graphics. Proceedings of SIBGRAPI VI, Recife, Brazil, 9–18.

Cornell, C. A. (1969). A probability-based structural code. Journal of the American Concrete Institute, 66(12), 974–985.

Cremona, C., & Gao, Y. (1997). The possibilistic reliability theory: theoretical aspects and applications. Structural Safety, 19(2), 173–201.

de Figueiredo, L. H., & Stolfi, J. (2004). Affine arithmetic: concepts and applications. Numerical Algorithms, 37(1–4), 147–158.

Elishakoff, I., & Ohsaki, M. (2010). Optimization and Anti-Optimization of Structures under Uncertainty. Imperial College Press, London.

Ferson, S., & Ginzburg, L. R. (1996). Different methods are needed to propagate ignorance and variability. Reliability Engineering and System Safety, 54(2–3), 133–144.

Ferson, S., Kreinovich, V., Ginzburg, L., Myers, D. S., & Sentz, K. (2015). Constructing Probability Boxes and Dempster-Shafer Structures. Sandia Report SAND2002-4015. Sandia National Laboratories, Albuquerque, NM.

Frangopol, D. M., Kallen, M. J., & van Noortwijk, J. M. (2004). Probabilistic models for life-cycle performance of deteriorating structures: review and future directions. Progress in Structural Engineering and Materials, 6(4), 197–212.

Hasofer, A. M., & Lind, N. C. (1974). Exact and invariant second-moment code format. Journal of Engineering Mechanics, ASCE, 100(EM1), 111–121.

JCSS (2001). Probabilistic Model Code. Joint Committee on Structural Safety, Zurich. Available at: www.jcss.byg.dtu.dk.

Möller, B., & Beer, M. (2004). Fuzzy Randomness: Uncertainty in Civil Engineering and Computational Mechanics. Springer, Berlin.

Moore, R. E. (1966). Interval Analysis. Prentice-Hall, Englewood Cliffs, NJ.

Nowak, A. S. (1995). Calibration of LRFD bridge design code. Journal of Structural Engineering, ASCE, 121(8), 1245–1251.

Nowak, A. S., & Collins, K. R. (2013). Reliability of Structures, 2nd Edition. CRC Press, Boca Raton.

Qiu, Z., & Elishakoff, I. (1998). Antioptimization of structures with large uncertain-but-non-random parameters via interval analysis. Computer Methods in Applied Mechanics and Engineering, 152(3–4), 361–372.

Qiu, Z., Ma, L., & Wang, X. (2009). Non-probabilistic interval analysis method for dynamic response analysis of nonlinear systems with uncertainty. Journal of Sound and Vibration, 319(1–2), 531–540.

Ministry of Roads and Bridges (MoRB) (2022). National Bridge Inspection Programme: Juba Ring Road Technical Report. Ministry of Roads and Bridges, Juba.

Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353.

Ang, A. H.-S., & Tang, W. H. (2007). Probability Concepts in Engineering: Emphasis on Applications in Civil and Environmental Engineering, 2nd Edition. John Wiley & Sons, New York.

Sofi, A., Romeo, E., Barrera, O., & Cocks, A. (2019). An interval finite element method for the analysis of structures with spatially varying uncertainties. Advances in Engineering Software, 128, 1–19.